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The use of the approximately normal distribution to describe the production of neutral particles

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Abstract. The normal distribution in two variables is used in an approximate form to describe the data on the reactions pp, π^-p or π^-n $\rightarrow$ (j negatively charged particles) + (neutral pions). It is shown that the model predicts a linear relationship between j and the mean number of neutrals. Modifications of the model are proposed in order to incorporate specific isospin conserving mechanisms. Comparisons are made with the work of other authors. Neutral strange particle production is discussed.

1. Introduction

There is now in existence a considerable body of data on the mean number of neutral pions $\langle n_0 \rangle_j$ produced in hadron–hadron collisions when it is known that j negatively charged particles have also been produced†. The most striking feature of these data is that plots of $\langle n_0 \rangle_j$ as a function of j are approximately linear (see figures 1 and 2).

A number of papers have appeared attempting to explain these data in terms of various isospin conserving mechanisms (Berger *et al* 1973, Horn and Schwimmer 1973, Drijard and Pokorski 1973). In this paper we intend to discuss a model that predicts the linearity of $\langle n_0 \rangle_j$ as a function of j for a large class of isospin conserving mechanisms. This model is the normal distribution in two variables

$$P(j, n_0) = \frac{1}{2\pi\xi_j\xi_0(1-\rho^2)^{1/2}} \exp\left(-\frac{(x^2 - 2\rho xy + y^2)}{2(1-\rho^2)}\right). \tag{1}$$

$P(j, n_0)$ is the probability that j negatively charged and n_0 neutral particles will be produced. Here $x = (j - m_j)/\xi_j$ and $y = (n_0 - m_0)/\xi_0 : m_j, \xi_j, m_0, \xi_0$ and ρ are energy dependent parameters.

† Data for pp interactions exist at the following values of P_{LAB} (the laboratory momentum): 12·4 GeV/c (Campbell *et al* 1973), 19 GeV/c (Scandinavian collaboration 1971), 70 GeV/c (French–Soviet collaboration 1973), 205 GeV/c (Charlton *et al* 1972), 303 GeV/c (Dao *et al* 1973). There are also results from the CERN ISR with $\sqrt{s}$ (the total energy) equal to 53 GeV (Flügge *et al* 1972). Here, however, the apparatus did not cover the full 4π geometry. For π^-p, data have been taken for P_{LAB} equal to 25 GeV/c (Elbert *et al* 1970) and 40 GeV/c (Bucharest–Budapest–Cracow–Dubna–Hanoi–Serpukhov–Sofia–Tashkent–Tbilisi–Ulan–Bator–Warsaw collaboration 1973a, b). In this last experiment data were also taken for π^-n interactions. In this paper we shall also require data on $\sigma(j)$, the cross section for the production of j negatively charged particles. These are to be found in the work of the Soviet–French collaboration (1972) (70 GeV/c pp interactions), Charlton *et al* (1972) (205 GeV/c interactions) and Dao *et al* (1972) (303 GeV/c interactions). In all other cases the data on $\sigma(j)$ and $\langle n_0 \rangle_j$ appear in the same paper with the exception of the ISR experiment for which there are no measurements of $\sigma(j)$.

The first authors to suggest the use of equation (1) were Parry and Rotelli (1973). We, however, shall do several things that these authors did not, as will become clear later in the paper.

A trivial calculation tells us that

$$\langle n_0\rangle_j = \sum_{n_0} n_0 P(j, n_0) \Big/ \sum_{n_0} P(j, n_0)$$
$$= m + \rho\xi_0(j - m_j)/\xi_j, \tag{2}$$

in qualitative agreement with the data. We also note that the dispersion of the neutrals, $(\langle n_0^2\rangle_j - \langle n_0\rangle_j^2)^{1/2}$ is given by $\xi_0(1-\rho^2)^{1/2}$, a constant. This feature of the model cannot be changed and, if measurements of the correlations between two π^0's and charged pions are ever made, will provide a severe test of equation (1).

In § 2 we shall present a brief derivation of the model. In § 3 we shall examine the approximations necessary in order to describe data at energies that are less than asymptotic and shall compare the model with the data that we have already listed, finding reasonable agreement.

In § 4 we shall use the model to describe data on the reaction

$$P_{\text{LAB}} \rightarrow (j \text{ negatives}) + \text{neutral strange particles} + \text{anything},$$

for which data exist with P_{LAB} equal to 205 GeV/c (Charlton *et al* 1973) and 303 GeV/c (Dao *et al* 1973). We shall find qualitative agreement.

In § 5 we shall discuss how to introduce an isospin conserving mechanism in a way that is consistent with the derivation of equation (1) given in § 2. We shall show that, as P_{LAB} becomes large, there is some evidence that a model in which most produced particles are one of a pair originating from the decay of an isoscalar resonance will describe the data, if we make the assumption that the production of these resonances is governed by the approximate normal distribution in one variable. In § 6 we shall discuss several alternative ways of incorporating isospin conservation, but without success.

In § 7 we shall show that there is some evidence that, when j is fixed, $\langle n_0\rangle_j$ is a function of the available energy only and that it does not depend on the initial state. Finally, in § 8, we shall summarize our conclusions and compare our work with that of other authors.

2. Derivation

The derivation of equation (1) is similar to that of the normal distribution in one variable alone as discussed by Kaiser (1972). The basic assumption is that, when two hadrons collide, they behave as if made up of a large, energy dependent number N of *independent* scattering centres. At each centre there is activity that may lead to the production of embryonic particles. In a final state interaction, these share the momentum carried by the individual centres, thus building real hadrons and determining their momentum distributions (a subject that will not be studied in this paper).

Let us suppose that at the νth centre j_ν negatively charged particles and n_{0_ν} neutrals are produced with probabilities dictated by some distribution $P_\nu(j_\nu, n_{0_\nu})$: using an obvious notation this has means m_{j_ν} and m_{0_ν}, variances ξ_{j_ν} and ξ_{0_ν} and covariance μ_ν.

We use the following definitions:

$$m_j = \sum_{\nu=1}^{N} m_{j_\nu} \qquad m_0 = \sum_{\nu=1}^{N} m_{0_\nu}$$

$$\xi_j^2 = \sum_{\nu=1}^{N} \xi_{j_\nu}^2 \qquad \xi_0^2 = \sum_{\nu=1}^{N} \xi_{0_\nu}^2$$

$$\rho = \left(\sum_{\nu=1}^{N} \mu_\nu\right)(\xi_j \xi_0)^{-1} \tag{3}$$

$$x = (j - m_j)/\xi_j$$

and

$$y = (n_0 - m_0)/\xi_0.$$

We may now use the two-dimensional analogue of the central limit theorem of probability (Cramér 1966) which tells us that, subject to very general restrictions on $P_\nu(j_\nu, n_{0_\nu})$, and neglecting terms of order $N^{-1/2}$, the distribution of produced pairs and neutrals is given by equation (1).

There are certain theoretical advantages to be gained by using equation (1). For example, the appearance of the variable x means that it does not matter whether we use j (the number of produced negative particles), $2j$ (the number of produced charged particles) or n_{ch} (the total number of charged particles in the final state), since x remains the same for each. If we were using the Poisson distribution (say) the choice of variable then becomes important (see Kaiser 1972 for a detailed explanation). Another advantage is that the steps of our argument follow irrespective of whether we specify pions only, kaons only, pions plus kaons, and so on. (In a more detailed model, of course, the values of the parameters will depend on the choice that we make.)

Another important feature of the model is that equation (1) should follow for any isospin conservation mechanism that does not violate the hypothesis that the scattering centres are independent. Clearly there is a large class of mechanisms for which this is true.

We also note that, integrating equation (1) over n_0 to give the charged particle multiplicity distribution $P(j)$, we find that

$$P(j) = \exp(-x^2/2)/\sqrt{2\pi}\xi_j \tag{4}$$

which is the normal distribution in one variable.

3. Application

Equation (1) cannot be used as it is because: (*a*) $j \geqslant 0$ and $n_0 \geqslant 0$, whereas x and y should extend from $-\infty$ to ∞; and (*b*) j and n_0 are discrete whereas x and y should be continuous. We make the simplest possible approximation by truncating equation (1):

$$P(j, n_0) = \exp[-(x^2 - 2\rho xy + y^2)/2(1-\rho^2)]/\Sigma' \qquad (j \geqslant 0 \text{ and } n_0 \geqslant 0)$$

where

$$\Sigma' = \sum_{j \geqslant 0} \sum_{n_0 \geqslant 0} \exp[-(x^2 - 2\rho xy + y^2)/2(1-\rho^2)]$$

and

$$P(j, n_0) = 0 \qquad (j < 0 \text{ and/or } n_0 < 0). \tag{5}$$

For a different procedure, see Parry and Rotelli (1973).

We now find that

$$P(j) = \frac{1}{\Sigma'} \exp(-x^2/2) \sum_{n_0=0}^{\infty} \exp[-(y-\rho x)^2/2(1-\rho^2)] \tag{6}$$

$$= \frac{1}{\Sigma'} \exp(-x^2/2) F(-\Delta_j) \tag{7}$$

where

$$\Delta_j = [m_0 + (\rho\xi_0/\xi_j)(j-m_j)]/\xi_0(1-\rho^2)^{1/2} \tag{8}$$

and, as ξ_j and ξ_0 become large, (so that j/ξ_j and n_0/ξ_0 may be treated as continuous variables)

$$F(-\Delta_j) \sim \sqrt{2\pi}\,\xi_0(1-\rho^2)^{1/2} \operatorname{erfc}(-\Delta_j). \tag{9}$$

Here erfc is the complementary error function. As Δ_j becomes large, $\operatorname{erfc}(-\Delta_j)$ becomes equal to one. It is clear that, for small values of j, $\operatorname{erfc}(-\Delta_j)$ may well differ substantially from one and hence $P(j)$ as given by equation (7) will be somewhat smaller than given by the approximately normal distribution ($P(j) = \exp(-x^2/2)/\Sigma$) used by Kaiser (1972, 1973). Similarly, in the same approximation as was used to derive equation (9)

$$\langle n_0 \rangle_j = \frac{1}{\Sigma'} \exp(-x^2/2) \sum_{n_0=0}^{\infty} n_0 \exp[-(y-\rho x)^2/2(1-\rho^2)]/P(j) \tag{10}$$

$$\sim m_0 + \frac{\rho\xi_0}{\xi_j}(j-m_j) + \frac{\xi_0(1-\rho^2)^{1/2}\exp(-\Delta_j^2/2)}{\sqrt{2\pi}\operatorname{erfc}(-\Delta_j)}. \tag{11}$$

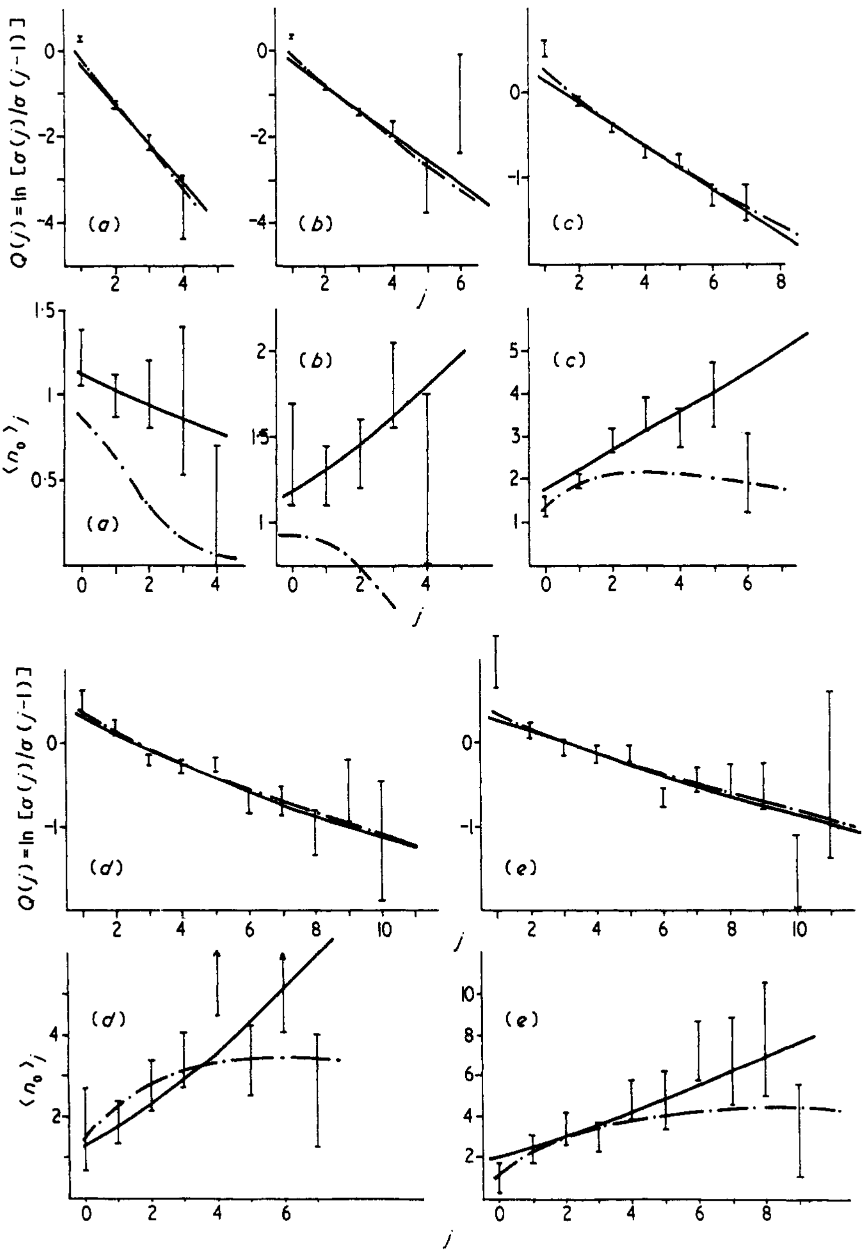
Deviations from the straight line equation (2) will occur for small j. The correction term is positive. As we shall see, in practice it turns out to be small for all values of j.

In order to discover whether the model agrees with the data quantitatively we have carried out a least squares fit. As explained by Kaiser (1973) we exclude $P(j = 0)$ (and hence $\langle n \rangle_{j=0}$) from our fitting routine because the model seems to require that we include in $P(j = 0)$ part (but not all) of σ(el), the elastic cross section. The parametrization of this contribution to σ(el) as a function of s is discussed by Kaiser (1973). We merely point out that its value tends to zero as $s \to \infty$. Presumably a similar parametrization could be used in this case if this were thought to be of sufficient interest. We also omit charge exchange processes in the π^-p case.

In figure 1 we have plotted both $Q(j) = \ln[\sigma(j)/\sigma(j-1)]$ (a convenient way of plotting the data) and $\langle n_0 \rangle_j$ as a function of j. The best fits using equations (5), (6) and (10), and renormalizing $P(j)$ to $cP(j)$ to take account of the omission of $P(j = 0)$ (so that c is another free parameter), are shown as full curves. In table 1 we present the values of χ^2 and the parameters for each energy.

We see that the values of χ^2 per degree of freedom are variable. The fits are good at 12·4 GeV/c, 19 GeV/c, 70 GeV/c and 300 GeV/c (for pp) and at 40 GeV/c for π^-p. When $P_{\text{LAB}} = 25$ GeV/c, the fit to the π^-p data is bad, due to the definite convex curvature of the data. When $P_{\text{LAB}} = 205$ GeV/c, agreement with the pp data is only moderate. The bulk of the disagreement stems from the values of $\langle n_0 \rangle_j$ when $j = 4$ and $j = 7$.

We conclude that agreement with data is satisfactory, considering the simplicity of the model that we are using. We note that corrections to $Q(j)$ and to $\langle n_0 \rangle_j$, which should be straight lines if the normal distribution holds exactly, are not very important—curvature of the full curves in figure 1 is just noticeable. As expected, extrapolating $Q(j)$ to $j = 1$ produces disagreement with the data, especially at low energies (note that σ(el) has been excluded from the experimental value of $\sigma(j = 0)$). It is by no means obvious, however, that the expected disagreement between the experimental and theoretical values of $\langle n_0 \rangle_{j=0}$ has materialized except in the case of the 40 GeV/c data. We note that



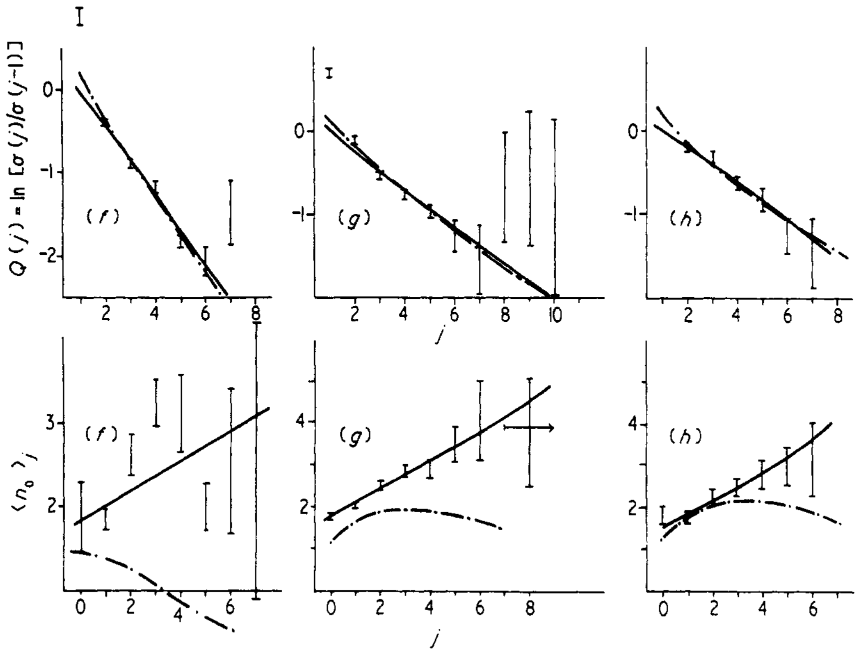


Figure 1. Plots of $Q(j) = \ln[\sigma(j)/\sigma(j-1)]$ and $\langle n_0 \rangle_j$ (defined in § 1) against j for the various data blocks. The full curves are the best fits from equations (5), (6) and (10). The chain curve arises from the combined use of binomial and normal distributions as in § 5. pp scattering at: (*a*) 12·4 GeV/*c*, (*b*) 19 GeV/*c*, (*c*) 70 GeV/*c*, (*d*) 205 GeV/*c*, (*e*) 303 GeV/*c*; π p scattering at: (*f*) 25 GeV/*c*, (*g*) 40 GeV/*c*; π^-n scattering at: (*h*) 40 GeV/*c*.

Table 1. Values of parameters and χ^2 per degree of freedom for fits using equation (5).

Initial state	Laboratory momentum (GeV/*c*)	m_j	ξ_j	m_0	ξ_0	ρ	χ_2 per degree of freedom
pp	12·4	0·13 ± 0·01	1·06 ± 0·01	0·85 ± 0·15	1·17 ± 0·24	−0·13 ± 0·11	2·56/3
pp	19	0·13 ± 0·01	1·30 ± 0·01	1·31 ± 0·18	0·40 ± 0·02	0·30 ± 0·10	4·68/5
pp	70	1·04 ± 0·05	2·00 ± 0·02	2·22 ± 0·18	1·03 ± 0·33	0·88 ± 0·02	13·7/10
pp	205	1·52 ± 0·05	2·66 ± 0·05	1·59 ± 0·24	2·81 ± 0·34	0·75 ± 0·07	20·0/12
pp	303	1·94 ± 0·10	2·98 ± 0·07	2·45 ± 0·34	3·27 ± 0·40	0·67 ± 0·01	17·6/15
π^-p	25	0·48 ± 0·01	1·54 ± 0·01	1·93 ± 0·12	0·69 ± 0·11	0·40 ± 0·01	26·1/9
π^-p	40	0·87 ± 0·03	1·89 ± 0·02	2·09 ± 0·07	1·10 ± 0·07	0·56 ± 0·04	9·77/10
π^-n	40	0·50 ± 0·07	2·18 ± 0·03	1·27 ± 0·10	1·90 ± 0·10	0·48 ± 0·06	10·87/7

at low energies, where we expect the biggest disagreement (see Horn and Schwimmer 1973), errors on $\langle n_0 \rangle_{j=0}$ are large. At higher energies, we expect that the model will tend to $P(j = 0$; inelastic) and that there will therefore be less difference here.

We also note that, as we would expect from the way that m_0 and ξ_0 are mixed together in equation (2), the data are not particularly sensitive to the values of these parameters (see table 1).

4. Neutral strange particles

As explained in § 2, the model equation (1) should apply to any two kinds of particles. Data exist on the reaction pp → (j negatively charged particles)+(neutral strange particles)+(anything). The mean number of strange neutrals $\langle n_s \rangle_j$ has been measured as a function of j for $P_{LAB} = 205$ GeV/c and $P_{LAB} = 303$ GeV/c (Dao *et al* 1973, Charlton *et al* 1973). $\langle n_s \rangle_j$ should be linearly related to j (equation (2)), or at least approximately so (equation (11)). We see from figures 2(b) and 2(c) that the data are certainly consistent with this predicted linearity, although they are not good enough to make a fit worthwhile.

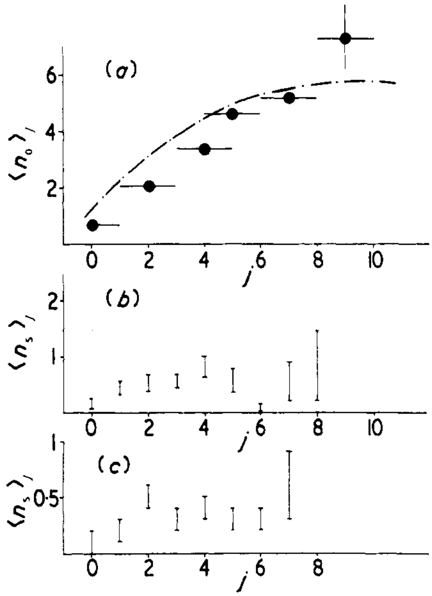


Figure 2. (a) $\langle n_0 \rangle_j$ against j(ISR). The chain curve is the fit using the model of § 5, $\sqrt{s} = 53$ GeV; (b) and (c) data on neutral strange particle production from experiments performed at NAL at 205 GeV/c and 303 GeV/c respectively.

5. Fewer parameters

It was pointed out in § 2 that one of the strengths of the theory is that equations (1) and (2) follow for a variety of isospin conserving mechanisms, and irrespective of whether certain kinds of particles (eg kaons) are included or excluded; that is, equations (1) and (2) are completely general.

This generality, however, is a weakness when it comes to making quantitative fits to the data since, at each value of P_{LAB}, we have five unrelated parameters m_j, ξ_j, m_0, ξ_0 and ρ. In what follows we want to attempt to produce a model which, while retaining the desirable features of the approximately normal distribution, has fewer parameters.

In order to produce a model that is consistent with our starting point, we need to preserve the independence of the scattering centres that we have introduced in § 2. This means that isospin must be conserved at each centre, and not imposed overall. The simplest assumption that we can make is that particles are produced from the decay of resonances of isospin zero. To make a more simplifying assumption, pions are produced in pairs, with a probability one third for a neutral pair and two thirds for a $\pi^+\pi^-$ pair. We assume that kaon and baryon pair production can be neglected. If the probability that n isoscalar resonances will be produced is $P_R(n)$, it follows that

$$P(j, n_0) = P_R(n)\binom{n}{j}(2/3)^j(1/3)^{n_0/2}. \tag{12}$$

It is clear that this model cannot be expected to work exactly since, for example, odd numbers of pions cannot be produced. If, however, the bulk of neutral pions is produced in pairs, it is plausible that, as s becomes large, equation (12) will approximate the data for $\langle n_0 \rangle_j$ in a reasonable manner.

So far, we have done no more than Drijard and Pokorski (1973), who investigated equation (12) under the name 'ϵ model'. Their principal result is that

$$\langle n_0 \rangle_j = (j+1)P(j+1)/P(j). \tag{13}$$

The value of this equation is that the relationship between $\langle n_0 \rangle_j$ and j can be investigated if we know the experimental values of $P(j)$. The new element that we introduce is a knowledge of $P_R(n)$: we assume that the production of the isoscalar resonances is described by the approximately normal distribution,

$$P_R(n) = \exp[-(n-m_R)^2/2\xi_R^2]/\Sigma_R$$

where

$$\Sigma_R = \sum_{n=0}^{\infty} \exp[-(n-m_R)^2/2\xi_R^2]. \tag{14}$$

Since for large enough n the binomial distribution may be approximated by a normal distribution, the combination of equation (14) and equation (12) will, for high energies (when we expect m_R and ξ_R to be large), give us equation (1) plus a knowledge of m_j, ξ_j, m_0, ξ_0 and ρ as functions of m_R and ξ_R, thus reducing the number of parameters.

On figure 1 we have displayed the best fit of equations (14) and (12) to $P(j)$ only (excluding $P(j = 0)$, as explained in § 3). Once we have done this we can use equation (13) to calculate $\langle n_0 \rangle_j$, also displayed on figure 1. We note the following features.

(i) The fits to $P(j)$ are extremely good; see table 2 for values of χ^2 per degree of freedom.

(ii) At lower values of P_{LAB}, the theoretical and experimental values of $\langle n_0 \rangle_j$ do not agree.

(iii) As P_{LAB} increases to 300 GeV/c, the theoretical values of $\langle n_0 \rangle_j$ are in much better agreement with experiment.

Table 2. Parameters and χ^2 per degree of freedom for the model of § 5.

Initial state	P_{LAB} (GeV/c)	m_R	ξ_R	χ^2 per degree of freedom
pp	12·4	0·88 ± 0·04	0·98 ± 0·03	0·6/1
pp	19	1·02 ± 0·03	1·26 ± 0·03	2·79/3
pp	70	2·36 ± 0·04	2·21 ± 0·03	4·18/5
pp	205	3·72 ± 0·10	2·96 ± 0·08	8·3/7
pp	303	4·23 ± 0·12	3·51 ± 0·12	11·2/9
π^-p	25	1·65 ± 0·02	1·50 ± 0·01	6·38/4
π^-p	40	1·89 ± 0·08	2·29 ± 0·03	10·56/7
π^-n	40	2·33 ± 0·12	2·26 ± 0·06	3·95/4

Our conclusion is that there is some evidence that the model described by equations (12) and (14) is able to describe the data well at high energies. There is also a plausible qualitative explanation as to why the model fits $P(j)$ well and $\langle n_0 \rangle_j$ not as well. We shall illustrate this explanation by constructing a simple idealized model.

Let us assume that the independent centres of which we wrote in § 2 are deposited in the centre of mass as a slowly moving fireball. There are also two leading particles. There is a probability q that one of these particles will remain a proton, and a probability $1-q$ that it will decay into a $p\pi^0$ pair. It follows that, if $P_R(n)$ is now the probability that n isoscalar resonances will be produced from the central fireball

$$P(j, n_0) = q^2 P_R(n) \binom{n}{j} (2/3)^j (1/3)^{n_0/2}$$

$$+ (1-q)^2 P_R(n-1) \binom{n-1}{j} (2/3)^j (1/3)^{(n_0/2)-1} \theta(n-1) \tag{15}$$

for n_0 even and $n = j + n_0/2$, and

$$P(j, n_0) = 2q(1-q) P_R(n) \binom{n}{j} (2/3)^j (1/3)^{(n_0-1)/2} \tag{16}$$

for n_0 odd and $j + (n_0-1)/2 = n$.

It follows that

$$P(j) = \sum_{n_0} P_R(n) \binom{n}{j} (2/3)^j (1/3)^{n_0/2}, \tag{17}$$

ie the expression for $P(j)$ is the same as that derived from equation (12). On the other hand,

$$P(j)\langle n_0 \rangle_j = \sum_{\substack{n_0 \\ \text{even}}} n_0 P_R(n) \binom{n}{j} (2/3)^j (1/3)^{n_0/2} + 2q(1-q)P(j) + 2(1-q)^2 P(j)$$

that is,

$$\langle n_0 \rangle_j = \langle n_0 \rangle_j^{(12)} + 2 - 2q \tag{18}$$

where $\langle n_0 \rangle_j^{(12)}$ is the mean derived from equation (12) alone. That is, we have an explicit model in which it is clear that $P(j)$ is exactly as derived from equations (12) and (14), whereas $\langle n_0 \rangle_j$ is somewhat greater than given by the fits shown on figure 1—a correction that is qualitatively in the right direction to improve agreement with experiment.

Variations on this sort of model are endless. For example, in the case of $\pi^- p$ scattering we can allow the possibility of charge exchange between the two leading particles, or we can allow more complicated decays of leading fireballs.

To conclude this section, we shall discuss the ISR data as displayed on figure 2(*a*). We have no data on $P(j)$ when $\sqrt{s} = 53$ GeV. On the other hand, Kaiser (1973) has discussed an energy dependent parametrization of $P(j)$ using the approximately normal distribution in one variable only and has shown that the model reproduces the data well when P_{LAB} takes on values in the range 5–300 GeV/c (for pp scattering). We assume that the parametrization can be extrapolated to $\sqrt{s} = 53$ GeV and use equation (13) to calculate $\langle n_0 \rangle_j$. The result appears on figure 2(*a*). Agreement is not perfect. On the other hand, the experimental results were not taken with a 4π geometry, and, of course, the extrapolation of the parametrization is of doubtful validity.

6. Alternative models

It is clear that the simple assumptions of § 5 are too crude to give any more than qualitative agreement with the data. Possible modifications might be to include isoscalar resonances decaying into more than two particles, or to assign charge and nonzero isospin to the scattering centres (a reasonable procedure if we wish to identify these centres with quarks). There is no obvious way in which we can proceed along these lines without introducing an excessive number of parameters, however.

Another way in which we could proceed is to assume that the total number N of produced particles follows an approximately normal distribution

$$P_T(N) = \exp[-(N-m)/2\xi^2]/\Sigma_N \tag{19}$$

where Σ_N is the usual normalization factor. We then violate the spirit of the derivation given in § 2 by assuming that isospin conservation is imposed in an overall way—for example, by assuming that when we have N pions and two nucleons, the probability that j of these are negatives and n_0 neutrals is given by the Cerulus statistical weight, $C(N; j, n_0)$, details of the calculation of which are given by Cerulus (1960). Briefly, the pair of particles pp (for example) is in an isospin $|11\rangle$ state. The combination (nucleon + nucleon + N pions) spans an isospin space that contains as a sub-space several orthogonal $|1, 1\rangle$ states. It is assumed that transitions to any of these states are equally likely. The calculation of the $C(N; j, n_0)$ is then merely a matter of calculating the weights to be attached to each orthogonal $|1, 1\rangle$ state for each combination of the values N, j and n_0. Hence

$$P(j, n_0) = P_T(N)C(N; j, n_0). \tag{20}$$

On figure 3 we show the 70 GeV/c data only, plotting both $Q(j)$ and $\langle n_0\rangle_j$ as a function of j. On this graph we have drawn the following: (i) the fit of § 2 (full curve); (ii) the fit of § 5 (chain curve); and (iii) the best fit of equations (19) and (2) (dotted curve). We note that $Q(j)$ is reproduced very well, but that $\langle n_0\rangle_j$ is too big for most of the values of j. These two features are characteristic of the model for all of the other data blocks. We also show: (iv) the fit of the Chow–Rix model (broken curve) in which equation (19) is replaced by a Poisson distribution before insertion into equation (20) (Chow 1970, Chow and Rix 1970). It is clear that this model does not reproduce the data.

Other models that we have tried include the replacement of equation (19) by the Feynman gas model of order 2 (Mueller 1971) which tells us that

$$P_T(N) = \exp(-F_1+F_2/2) \times\left[\frac{(F_1-F_2)^j}{j!}+\frac{(F_1-F_2)^{j-2}}{(j-2)!}\frac{F_2}{2}+\ldots\frac{(F_1-F_2)^{j-2k}}{(j-2k)!k!}\left(\frac{F_2}{2}\right)^k+\ldots\right]. \tag{21}$$

This, combined with equation (20), leads to a fit approximately like that drawn as a dotted curve in figure 3. The Chew–Pignotti model (Chew and Pignotti 1968) which in essence uses the Poisson distribution for $P_T(N)$ together with a prescription for $C(N; j, n_0)$ which involves the exchange of alternate isospin zero and isospin one reggeons along a multiperipheral chain, produces a fit to $Q(j)$ similar to that of the Chow–Rix model.

By stepping outside the original assumptions of § 2, then, we find that it is not easy to improve agreement with experiment.

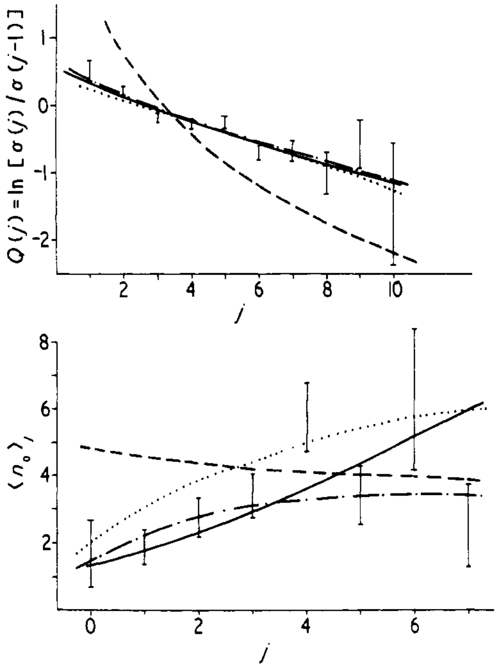


Figure 3. $Q(j)$ against j and $\langle n_0 \rangle_j$ against j for the 70 GeV/c pp data, with fits obtained by using various models: full curve, approximately normal distribution in two variables (§ 2); chain curve, isoscalar resonance model (§ 5); dotted curve, approximately normal distribution with Cerulus weights; broken curve, Chow–Rix model.

7. $\langle n_0 \rangle_j$ in pp and π^-p scattering

Kaiser (1974) used the normal distribution in one variable to describe charged particle multiplicity distributions in both pp and π^-p scattering, ie

$$P(j) = \exp[-(j-m)^2/2\xi^2]. \tag{22}$$

We wrote down simple, energy dependent parametrizations for m and ξ and showed that the data are such that, as s increases, $P(j)$ appears to become independent of the initial state; that is, m in pp and π^-p scattering becomes the same function of s as $s \to \infty$. A similar statement applies to ξ. An extension of this is that we expect $\langle n_0 \rangle_j$ to become independent of the initial state as $s \to \infty$.

In figure 4 we have plotted $\langle n_0 \rangle_j$ as a function of $(E_{\mathrm{av}})^{-1}$ for various values of j, omitting $j = 0$ as we have done all through this paper. E_{av} is the available energy, $\sqrt{s}$ —the sum of masses of particles in the initial state. We have chosen $(E_{\mathrm{av}})^{-1}$ to provide a scale on which the experimental points are reasonably spaced. The data are well consistent with the hypothesis that, for fixed j, $\langle n_0 \rangle_j$ is a function of the available energy only. (The errors on the data are such that we ought not to be too firm in concluding this, however.)

Note that, for fixed j, the data suggest that $\langle n_0 \rangle_j$ rises as E_{av} increases. This is a conclusion that differs from that of the French–Soviet Union collaboration (1973) where it is tentatively concluded, by studying the 70 GeV/c and 205 GeV/c data only, that $\langle n_0 \rangle_j$ may become constant as $s \to \infty$.

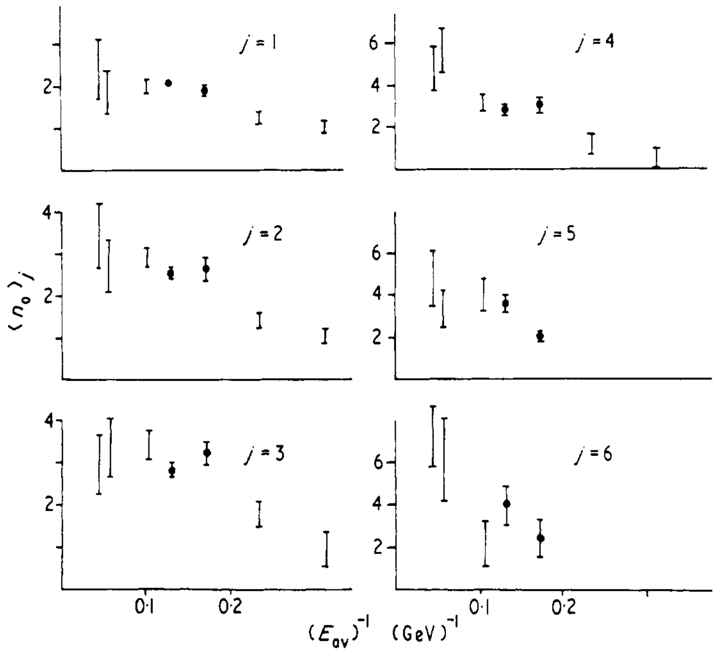


Figure 4. $\langle n_0 \rangle_j$ against $(E_{av})^{-1}$ for various values of j. I, pp scattering; ⧫, $\pi^- p$ scattering.

8. Conclusion

We have shown that the approximately normal distribution in two variables (equation (5)) is in reasonable agreement with the data on $\langle n_0 \rangle_j$ and $P(j)$ and that the model is in agreement with what data there is on neutral strange particle production. We have also shown that there is a simple derivation of equation (1) that is a natural extension of that used for the normal distribution in one variable as described by Kaiser (1972). In an attempt to reduce the number of parameters, it was demonstrated that there is some evidence that a model in which isoscalar resonances decay into pion pairs describes the data as s becomes large, and that several other simple attempts to achieve the desired reduction fail.

As we have said, Parry and Rotelli (1973) are the authors who first proposed the use of equation (1) to describe the simultaneous production of two kinds of particle. Our work differs from theirs in several respects. We have given a derivation of equation (1). We have made an effort to assess the success of the model quantitatively. We have used a different truncation procedure (equation (5)). We have discussed ways of reducing the numbers of parameters. Finally, we have compared the model to data on neutral strange particle production.

Our work also differs from that of other authors who consider the production of neutral particles (Berger *et al* 1973, Horn and Schwimmer 1973, Drijard and Pokorski 1973). In all cases, these papers deal with specific isospin conserving mechanisms, whereas we have demonstrated that, given the independence of scattering centres as described in § 2, the linearity of $\langle n_0 \rangle_j$, as a function of j follows irrespective of the isospin conserving mechanism. In considering the model of § 5, there are similarities to the ϵ model of Drijard and Pokorski (1973). These authors, however, are concerned with the information about $\langle n_0 \rangle_j$ that can be gained from a knowledge of $P(j)$ only whereas we are interested in the study of the normal distribution (equation (14)).

Other authors (Berger *et al* 1973, Horn and Schwimmer 1973) investigate models in which, first of all, certain resonances (σ or ρ, say) are produced by Poisson-like mechanisms, or by inverse power law mechanisms which follow from fragmentation models. The decay of these resonances then provides a way of calculating $\langle n_0 \rangle_j$. Alternatively, pions are produced by Poisson-like or fragmentation type mechanisms and isospin is conserved in an average way. In short, our approach is distinguished by the use of the approximately normal distribution, which is not considered by any of the authors mentioned in this paragraph. (Horn and Schwimmer (1973), however, do point out that the gaussian distribution in two variables is a good approximation to some of their models.)

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